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Equivariant discrete Morse theory

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ABSTRACT

In this paper, we study Forman's discrete Morse theory in the case where a group acts on the underlying complex. We generalize the notion of a Morse matching, and obtain a theory that can be used to simplify the description of the G-homotopy type of a simplicial complex. As an application, we determine the $C_2 \times S_{n-2}$-homotopy type of the complex of non-connected graphs on n nodes.

1. Introduction

This paper has two major goals, one general and one more specific. The first goal is to extend the ideas of discrete Morse theory from the category of simplicial complexes to the category of simplicial G-complexes. That is, given a group action by G on a simplicial complex Σ, we want to construct a complex $\mathcal{M}$ with $\mathcal{M} \simeq_G \Sigma$, where $\mathcal{M}$ will ideally be a much smaller complex than Σ. This is non-trivial, since it is hard to construct a reasonably big G-invariant matching on the face poset $P(\Sigma)$. The problem is relevant since the simplicial complexes arising in practice often are naturally equipped with some group action, which is an intrinsic part of its structure.

The complex $\mathcal{M}$ respects all G-homotopy invariants of Σ. Examples of such invariants are the G-module structure on the homology of Σ, the homotopy type of the space Σ^G of fixed points, or of the quotient space Σ/G.

To solve the problem, we introduce a generalization of the notion of a Morse matching. Then, we use such a generalized Morse matching together with certain linear collapses of simplices, to obtain a G-equivariant homotopy equivalence $\mathcal{M} \simeq_G \Sigma$.

The second goal is to solve a problem in graph theory, namely to calculate the G-homotopy type of a simplicial complex $\mathcal{N}_n$, determined by the set of non-connected graphs on n nodes. Here, $G = C_2 \times S_{n-2}$, and the G-action is induced by a natural action of G (as a subgroup of S_n) on the n nodes.

As a result, we can calculate the G-action of the homology of $\mathcal{N}_n$.

2. Preliminaries

We assume familiarity with the basic notions of Discrete Morse theory, as presented in [3,2]. For general questions about G-homotopy, we refer to [8].

If σ is an abstract simplex, we denote its geometric realization $|\sigma|$. Similarly, if Σ is an abstract simplicial complex, then $|\Sigma|$ is its realization. We will restrict our attention to finite simplicial complexes, and we will omit the word "finite" when referring to them.

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$$\begin{tikzcd} G \times_{H_\alpha} \partial\Delta^m_\alpha \arrow[r, "\Phi_\alpha"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\ulcorner", very near start] & X_{n-1} \arrow[d] \\ G \times_{H_\alpha} \Delta^m_\alpha \arrow[r] & X_n \end{tikzcd}$$

Fig. 1. The construction of X_n from X_{n-1}.

If σ is a maximal proper face of τ, we will say that σ is a *facet* of the simplex τ, for short. This should not be confused with the notion of a facet of a simplicial complex.

We define a *simplicial G-complex* to be a simplicial complex Σ, together with a group action by G. In the standard setting, the Morse complex is a CW-complex. The "G-analogue" of these will be *GCW-complexes modelled by twisted products*, constructed as follows:

Definition 2.1. Let G be any group, and let H be a subgroup of G acting on a space X. Regard G as a topological space by giving it the discrete topology. Define an equivalence relation $\sim$ on $G \times X$ by $(gh, x) \sim (g, hx)$.

We define the space $G \times_H X$ to be the quotient space $G \times X/\sim$, and we call it a ***twisted product***.

G acts on $G \times_H X$ by $g(g', x) = (gg', x)$

In a category theorist's language, we have freely constructed a G-space, given an H-space.

Example 2.1. The discrete space on three points with its natural S_3 action, is $S_3 \times_{(i)} \{x\}$, where $i \in S_3$ is an involution acting trivially on $\{x\}$.

Indeed, $S_3 \times \{x\}$ is the discrete space on six points, and in $S_3 \times_{(i)} \{x\}$ we have identified the points $(r^k i, x)$ with $(r^k, ix) = (r^k, x)$, where $r \in S_3$ is a rotation. Clearly then, r rotates the three points of $S_3 \times_{(i)} \{x\}$, and i interchanges the points (r, x) and (r^2, x).

Next, we will construct, for a fixed group G, complexes with twisted products as building blocks. This is done inductively, as follows:

1. Start with a discrete set X_0 of points, and an action on X_0 by G.
2. Given X_{n-1}, attach twisted products $G \times_{H_\alpha} \Delta^m_\alpha$ to X_{n-1} via G-equivariant inclusions of their boundaries. We then obtain X_n. The H_α's are subgroups of G.

   Here, Δ^m_α are m-simplices (the m's possibly different from each other and from n), acted "linearly" upon by H_α. Formally, we could say that Δ^m_α is a disc of a representation of H_α. Now, α indexes all cells added to X in the nth stage.

   It is as trivial as it is important to check that $G \times_{H_\alpha} \partial\Delta^m_\alpha = \partial(G \times_{H_\alpha} \Delta^m_\alpha)$ (see Fig. 1).
3. We have natural inclusions $X_0 \subseteq X_1 \subseteq \cdots$. Define $X = \bigcup_n X_n$. We will call X a GCW-complex modelled on discs of representations.

Remark 1. This is analogous to the standard construction of ordinary CW-complexes, as in for example [5]. However, when we construct CW-complexes, we usually assume $n = m$, so we always add cells to cells of lower dimensions. For the G-equivariant collapses in chapter 6 to work out, we will have to allow $n \neq m$. We also need to allow internal action on the cells, which is why we model the complexes on twisted products.

A simplicial G-complex can be regarded as a GCW-complex in a natural way, by adding its cells in increasing order of dimension, and letting H_α be the subgroup of G under which Δ_α is invariant. The complex will then have one Δ^n_α for each orbit of n-simplices.

Any GCW-complex also has an underlying CW-complex, since when the group action is ignored, the cell $G \times_H \Delta^m$ is homeomorphic to $[G : H]$ copies of Δ^m. When talking about (the number of) cells of a GCW-complex, we will actually mean cells in this underlying CW-complex.

3. Generalized Morse matchings

We start by presenting a concept of generalized Morse matchings. These can be viewed either as a proper generalization, or as a decomposition of a Morse matching into an 'internal' and an 'external' part. We will later see how this helps us make matchings that are invariant under group actions. Heuristically, we will make the external part of the matching invariant, and then handle the internal part by other methods, not properly belonging to discrete Morse theory.

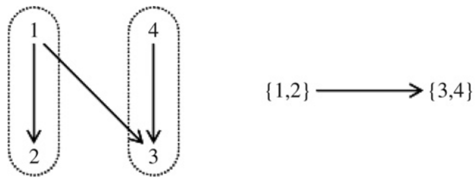

Fig. 2. G, with the matching $\sim$ indicated by dotted lines. On the right, $G/\sim$.

3.1. Classical Morse matchings

Given the simplicial complex Σ, consider the inclusion poset of its faces $P(\Sigma)$. We will view this as a directed graph, where the arrows indicate the cover relations. Recall that τ *covers* σ if $\sigma < \tau$ and there is no element ρ with $\sigma < \rho < \tau$. Note that this is equivalent to σ being a facet of τ in Σ. If this is the case, we draw an arrow $\tau \to \sigma$ in the graph.

Definition 3.1. Given a directed graph $G = (V, A)$ and an equivalence relation $\sim$ of its vertices, construct the graph $G/\sim$ as follows: The vertex set of $G/\sim$ is $V/\sim$, and there is an arc from v_1 to v_2 if there are $w_1, w_2 \in V$ with $w_i \in v_i$, and an arc $(w_1, w_2) \in A$.

As a special case, we can let the equivalence relation be given by a matching of the vertices, as in Fig. 2.

It is well known that a matching M on $P(\Sigma)$ is a discrete Morse matching iff $P(\Sigma)/M$ has no directed loops. In this case, there is a Morse complex $\mathcal{M} \simeq \Sigma$, whose cells are in one-to-one correspondence with the unmatched cells in Σ.

3.2. Generalized Morse matchings

This inspires us to think further. Could these constructions be carried out on $P(\Sigma)/\sim$ for some partition $\sim$ that is not a matching? As a matter of fact, they can. We will let $\sim$ be a partition of $P(\Sigma)$ into intervals, and impose the same restriction as before on $P(\Sigma)/\sim$. The standard case occurs when every interval in the partition has rank 1 or 2, so $\sim$ is a matching M.

Any non-trivial interval $[\sigma, \tau]$ in $P(\Sigma)$ allows a complete acyclic matching. For example, we can choose $v \in \tau \setminus \sigma$ and match ρ to $\rho \cup \{v\}$ for all ρ not containing v.

Thus the following lemma follows as an immediate consequence of the cluster lemma, as presented in [7] or [6].

Lemma 3.1. *Let $\sim$ be an equivalence relation on Σ whose equivalence classes are intervals in $P(\Sigma)$. Suppose $P(\Sigma)/\sim$ is acyclic. Then there is a discrete Morse matching on Σ whose critical cells are exactly those that are alone in their equivalence classes under $\sim$.*

Let us call an equivalence relation $\sim$ that satisfies the assumptions in Lemma 3.1 a *generalized Morse matching*. This will hopefully cause no confusion, even though $\sim$ is no matching at all in a graph-theoretic sense.

Note that, while every generalized Morse matching can be refined to a standard Morse matching, it cannot be refined *in a natural way*, so it is quite possible and common that the generalized Morse matchings have nice properties that none of its refinements have. In particular, it will in general be much easier to find generalized matchings that are group invariant, than to find classical matchings.

4. Consequences for simplicial G-complexes

We have now constructed a generalization of the notion of a Morse matching. In this section, we will show how to use this generalization to obtain a discrete Morse theory for simplicial G-complexes.

4.1. Linear collapses

We will need two more definitions about simplicial complexes before getting started.

Definition 4.1. Let σ be a proper face of a simplex τ. The dual of σ with respect to τ is defined to be $\sigma_\tau^* \overset{\text{def}}{=} \tau \setminus \sigma$.

Remark 2. Here, σ and τ really refer to the abstract simplices, rather than their realizations. Hence, σ_τ^* is a face of τ, and σ and σ_τ^* together contain all vertices of τ.

Definition 4.2. Let σ be a proper face of τ. Define $P_\tau(\sigma)$ to be the union of all facets of τ that contain σ.

We see that, given a generalized Morse matching $\sim$ that contains $[\sigma, \tau]$, none of the cells in $P_\tau(\sigma)$ will be in the Morse complex induced by $\sim$. On the other hand, $P_\tau(\sigma)$ contains all facets of τ that are in $[\sigma, \tau]$. So we will want to collapse τ along $P_\tau(\sigma)$, onto $P_\tau(\sigma_\tau^*)$. The next proposition shows how we can do this in a nice way.

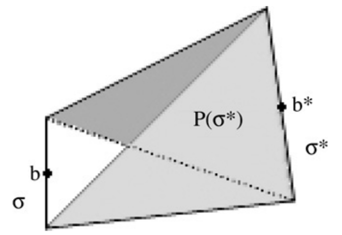

Fig. 3. An illustration where $\dim(\tau) = 3$, and $\dim(\sigma) = 1$.

Proposition 4.1. *Let σ be a proper face of τ. Then τ collapses onto $P_\tau(\sigma_\tau^*)$.*

Moreover, let g be a permutation of the vertices of τ, with $g(\sigma) = \sigma$. Extend g linearly to the geometric simplex $|\tau|$.

Then, the collapse $c : |\tau| \times I \to |\tau|$ can be chosen so that

$c(x, 0) = x$, $c(x, 1) \in P_\tau(\sigma_\tau^*)$ and $gc(x, s) = c(gx, s)$ for any $x \in \tau$ and $0 \leq s \leq 1$

Proof. Denote by b the barycenter of σ, and by b^* the barycenter of σ_τ^*. We claim that every point $x \in |\tau|$ can be written as $x = tb + (1-t)x^*$ where $x^* \in P_\tau(\sigma_\tau^*)$ and $0 \leq t \leq 1$ (see Fig. 3).

But every point in $|\sigma|$ can be written in this way, as $\partial\sigma \subseteq P_\tau(\sigma_\tau^*)$. Clearly every point in $|\sigma_\tau^*|$ can too, as $\sigma_\tau^* \subseteq P_\tau(\sigma_\tau^*)$. But then every convex combination of points in $|\sigma|$ and $|\sigma_\tau^*|$ can be written in this way, and the claim is proved since $|\sigma|$ and $|\sigma_\tau^*|$ span $|\tau|$ convexly.

Then, define $c(x, s) = t(sb^* + (1-s)b) + (1-t)x^*$. In words, c is defined by b being collapsed onto b^*, and the collapse being linear.

Now we have $c(x, 0) = tb + (1-t)x^* = x$ and $c(x, 1) = tb^* + (1-t)x^*$. This later point is a convex combination of two points in the $(n-1)$-cell containing x^*, since $x^* \in P_\tau(\sigma_\tau^*)$. So $c(x, 1)$ is in this same $(n-1)$-cell, and so is in $P_\tau(\sigma_\tau^*)$.

Finally, $gc(x, s) = t(sg(b^*) + (1-s)g(b)) + (1-t)g(x^*)$ by linearity of g. But since g only permutes the vertices of σ_τ^* internally, it leaves the barycenter fixed, $g(b^*) = b^*$ and so $gc(x, s) = t(sb^* + (1-s)g(b)) + (1-t)g(x^*) = c(gx, s)$.

This proves the proposition. □

4.2. Equivariant collapses to the Morse complex

It is worth noting that the collapse described above brings the barycenter of σ to the barycenter of σ_τ^*. Hence, if we apply this collapse to a complex where some cell is glued to σ, this cell might after the collapse be glued to cells of higher dimensions. This is why we had to accept $n \neq m$ when constructing our GCW-complexes modelled on discs of representations.

We will now see how a generalized Morse matching yields a collapse from Σ to the equivalent of the Morse complex.

Theorem 4.2. *Let $\sim$ be a generalized Morse matching on a finite simplicial G-complex Σ. Assume $\sim$ is G-equivariant, i.e. if $[\sigma, \tau]$ is in $\sim$, then so is $[g\sigma, g\tau]$.*

Then there is a GCW-complex $\mathcal{M}$, such that the cells of $\mathcal{M}$ (regarded as a CW-complex) are exactly the cells in Σ that are critical with respect to $\sim$, and $\Sigma \simeq_G \mathcal{M}$.

We will construct a collapse of Σ explicitly. This will turn out to be G-equivariant and the remaining complex will be seen to have the properties of $\mathcal{M}$. We will call the remaining complex the *Morse complex*, in analogy with the classical case.

Proof. By definition, $P(\Sigma)/\sim$ is acyclic, so it can be regarded as a poset, with $w > v$ if there is a path from w to v.

Since $\sim$ is G-equivariant, G acts on $P(\Sigma)/\sim$, and since $P(\Sigma)/\sim$ is finite, we cannot have $v < gv$ for any $v \in P(\Sigma)/\sim$, $g \in G$. So $v < w \Rightarrow gw \not< hv$ for $g, h \in G$, $v, w \in P(\Sigma)/\sim$. Hence, we can construct the poset $G \setminus (P(\Sigma)/\sim)$ by $G[\sigma] < G[\tau]$ iff $g[\sigma] < h[\tau]$ for some $g, h \in G$.

This poset has a linear extension $v_0 < v_1 < \cdots < v_N$. Keep in mind that each v_i is an orbit of intervals $[\sigma, \tau]$ in $P(\Sigma)$.

Now if σ is contained in v_j, then any face ρ of σ has $\rho < \sigma$ in $P(\Sigma)$, so ρ must be in v_i for some $i \leq j$. Hence for any n,

$$\Sigma_n \stackrel{\text{def}}{=} \bigcup_{k \leq n} \bigcup_{\sigma \in v_k} \sigma$$

is a G-subcomplex of Σ.

We will inductively construct the homotopy equivalences $\Sigma_k \simeq_G \mathcal{M}_k$ for some complex $\mathcal{M}_k$, constructed from those cells σ such that $v_i = G\{\sigma\}$ for some $i \leq k$. This will be done by putting certain collapses together.

Every interval has only one minimal element, so for every non-trivial interval, there is some element of $P(\Sigma)/\sim$ that is smaller. Thus the intervals in v_0 contain only one 0-cell, which is critical. So let $\mathcal{M}_0 = \Sigma_0$, and let the collapse be trivial.

Now assume that we have constructed the collapse of Σ_{m-1}. If $v_m = G\{\sigma\}$, the $g\sigma$:s are critical, and so

$$\Sigma_m = \Sigma_{m-1} \cup_{G \times_H \partial\sigma} (G \times_H \sigma) \simeq_G \mathcal{M}_{m-1} \cup_{G \times_H \partial\sigma} (G \times_H \sigma) \stackrel{\text{def}}{=} \mathcal{M}_m,$$

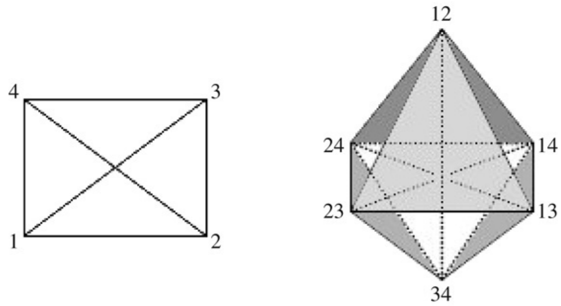

Fig. 4. The complete graph K_4, and the simplicial complex $\mathcal{N}_4$.

where the homotopy extends the homotopy $\Sigma_{m-1} \simeq_G \mathcal{M}_{m-1}$.

So assume $v_m = G[\rho, \sigma]$, where $\rho \neq \sigma$. Since $\sim$ is G-equivariant, any group element fixing σ must also fix ρ. So we can apply the collapse referred to in Proposition 4.1 for each interval in v_m, and get a collapse $\Sigma_m \searrow \Sigma_{m-1} \simeq_G \mathcal{M}_{m-1}$.

Since $\Sigma = \Sigma_n$ for some n, we get $\Sigma \simeq_G \mathcal{M}_n \stackrel{\text{def}}{=} \mathcal{M}$, and $\mathcal{M}$ is a complex with one p-cell for each critical p-simplex in Σ with respect to $\sim$. This completes the proof. $\square$

Finally, we quickly and informally discuss why generalized Morse matchings suit the equivariant case better than classical ones.

In a classical Morse matching, a simplex σ always has to be matched with a single facet $\tau = \sigma \setminus \{v\}$ of σ. If some subgroup $H \subseteq G$ permutes the vertices of σ, there may be no facet of σ that is invariant under H, though σ is. Then, we cannot match σ equivariantly, so σ has to be critical.

However, in a generalized Morse matching, we can match σ to any face ρ of σ, so the codimension does not have to be one. Hence, if H is σ's group of invariance, we can match σ whenever some **face** of σ is H-invariant, instead of only when some **codimension one face** is invariant. This clearly gives us way more opportunities.

5. An application to graph theory

In Section 5 of [4], Forman uses discrete Morse theory to determine the homotopy type of the complex of non-connected graphs, described below. In this section, we show how our equivariant discrete Morse theory strengthens these results, to determine the Γ-homotopy type of the complex, for a certain group Γ. In this section, we denote groups by Γ, leaving G to denote graphs.

5.1. The complex of non-connected graphs

We will denote by $\mathcal{N}_n$ the complex of non-connected graphs on n vertices, constructed as follows:

Consider the complete graph K_n on n labelled vertices, and let E_n be its edge set. Let $\mathcal{N}_n$ have vertex set E_n, which means $\mathcal{N}_n$ has $\binom{n}{2}$ vertices, one for each edge in K_n. Now every spanning subgraph of K_n (i.e. those subgraphs containing all n vertices) will naturally correspond to a subset of E_n.

Let the simplices of $\mathcal{N}_n$ be those subsets of E_n that correspond to non-connected graphs. Then $\mathcal{N}_n$ is a simplicial complex, since the property of being non-connected is closed under subgraphs.

Remark 3. The complex $\mathcal{N}_n$ is denoted Δ_n^1 in many sources, including [1,7,9]

Remark 4. The above construction works when non-connectedness is replaced by any property that is closed under subgraphs. Examples of such properties are k-colourable graphs, not k-connected graphs and acyclic graphs.

The complex $\mathcal{N}_4$ is shown in Fig. 4, and the reader may verify that its simplices exactly correspond to the non-connected subgraphs of K_4. You may also verify that the biggest non-connected graphs are isomorphic to K_{n-1} together with one isolated node. So in general, $\mathcal{N}_n$ has dimension $\binom{n-1}{2} - 1$.

Forman [4] uses discrete Morse theory to determine the homotopy type of $\mathcal{N}_n$. This result had earlier been found by Vassiliev using elementary methods in [10]. The result is that $\mathcal{N}_n \simeq \bigvee_{i=1}^{(n-1)!} S_i^{n-3}$, where $\bigvee_{i=1}^{k} S_i^n$ denotes the wedge of k spheres of dimension n (i.e. the CW-complex consisting of one 0-cell and k n-cells).

Forman does this by constructing a Morse matching whose Morse complex $\mathcal{M}_f$ consists of the cells corresponding to:

- The graph consisting of the single edge $\{1, 2\}$.

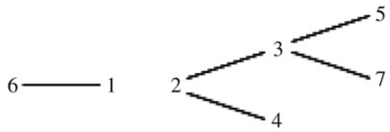

Fig. 5. A critical graph, according to Forman.

- The graphs with exactly two connected components, each of the components being a tree, rooted at 1 and 2 respectively. Moreover, the labels on each branch are increasing away from the roots. An example of such a graph is given in Fig. 5.

For $k = 3, \ldots, n$, the predecessor of k can be chosen in $k - 1$ different ways. So there are $(n-1)!$ such graphs, and clearly each of them has $n-2$ edges (and so constitute an $(n-3)$-cell of $\mathcal{N}_n$). It follows that $\mathcal{N}_n \simeq \bigvee_{i=1}^{(n-1)!} S_i^{n-3}$. For details, see [4].

5.2. Equivariant collapse of $\mathcal{N}_n$

Obviously, the property of being non-connected is a *graph property*, which means that it only depends on the isomorphism type of the graph. So any relabelling of the nodes of K_n will induce an automorphism of $\mathcal{N}_n$. Since a relabelling of the nodes is given by a permutation of $\{1, \ldots, n\}$, we have a group action by S_n on $\mathcal{N}_n$.

However, this group action does not restrict to the Morse complex $\mathcal{M}_f$ in the last section, since if the labels on the branches of G increase away from 1 and 2, this may not be so after a relabelling of the nodes. Geometrically, this means that the homotopy equivalences Forman constructs do not respect the symmetries of $\mathcal{N}_n$.

We would like to construct a Morse matching that respects the symmetries of $\mathcal{N}_n$. However, we do not want the Morse complex to be too big, so if we could do with one 0-cell and $(n-1)!$ cells of dimension $n-3$, that would be great, since that is the minimal CW-complex homotopy equivalent to $\bigvee_{i=1}^{(n-1)!} S_i^{n-3}$. We could not do that S_n-equivariantly, since there is no 0-cell in $\mathcal{N}_n$ that is fixed by S_n. So we will look at a slightly smaller group action.

Consider the subgroup $\Gamma \subseteq S_n$ that fixes $\{1, 2\}$. This group permutes the set $\{1, 2\}$ and it is clear that $\Gamma \cong C_2 \times S_{n-2}$, since Γ permutes $\{1, 2\}$ and $\{3, \ldots, n\}$ independently. So we have an action on $\mathcal{N}_n$ by $C_2 \times S_{n-2}$.

This is, in fact, the biggest group Γ such that we could possibly have $\mathcal{N}_n \simeq_\Gamma \mathcal{M}$, if $\mathcal{M}$ is isomorphic to $\bigvee_{i=1}^{(n-1)!} S_i^{n-3}$ as a CW-complex. This is because any such group must fix the unique 0-cell of $\bigvee_{i=1}^{(n-1)!} S_i^{n-3}$.

We will construct a generalized Morse matching $\sim$, whose Morse complex is isomorphic (as a CW-complex) to $\bigvee_{i=1}^{(n-1)!} S_i^{n-3}$, and such that $\sim$ is Γ-equivariant. We will then check how Γ acts on the Morse complex, and so determine the Γ-homotopy type of $\mathcal{N}_n$.

Let M denote the subset of $\mathcal{N}_n$ containing:

- The graph consisting of the single edge $e = \{1, 2\}$.
- The graphs that are the union of two chains, the node 1 being an endpoint on one of them, 2 being the endpoint of the other.

There are $(n-1)!$ graphs of the second kind, as the first chain can have any length between 1 and $n-1$, and the numbers $3, \ldots, n$ can be arranged in $(n-2)!$ ways in the graph.

Our goal is that $\sim$ should match everything except M. The matching $\sim$ will be constructed in two stages:

1. If the graph G contains the edge $e = \{1, 2\}$ and at least one more edge, match $[G \setminus e, G]$ in $\sim$. The graphs in $\mathcal{N}_n$ that are still unmatched are the graph with the single edge e, and the graphs G not containing e, but such that $G \cup e$ is connected. It is readily seen that these are the graphs that have two connected components, one containing 1, the other containing 2.
2. Let G be a graph that is not yet matched. Denote the connected components of G by G_1 (for the one containing 1), and G_2 (the one containing 2).

   Follow G_1 away from 1. If G_1 is not a chain, there is a first node v_1 at which this cannot be done uniquely. In other words, $v_1 \in G_1$ has degree $d(v_1) > 1$ if $v_1 = 1$ and $d(v_1) > 2$ if $v_1 \neq 1$, and among all such nodes, v_1 is the one with minimal distance to 1. Let S_1 be the set of neighbours to v_1, minus the one on the chain from v_1 to 1 (if $v_1 \neq 1$). If G_1 is a chain, set $S_1 = \emptyset$.

   If G_2 is not a chain, define v_2 and S_2 analogously, replacing 1 by 2.

   A picture will clarify things (see Fig. 6):

   Among the graphs unmatched in step 1, it is clear that $G \notin M$ iff at least one of v_1 and v_2 is defined, and then at least one of S_1 and S_2 has multiple elements.

   Let $G = G_1 \cup G_2$ be a not yet matched graph, that has no edges between nodes in S_1 or S_2. Construct from G the graph G', where you have added all possible edges within S_1 and S_2 (so that G' restricted to S_i is a complete graph). See Fig. 7.

   Match $[G, G']$ in $\sim$. Of course, $[G, G']$ is the set of all graphs H such that $G \subseteq H \subseteq G'$.

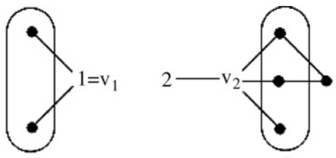

Fig. 6. A non-connected graph, unmatched after the first stage.

Fig. 7. A pair of graphs G and G', as described above.

Lemma 5.1. *The graphs in $\mathcal{N}_n$ left unmatched by $\sim$ are those in M.*

Proof. The graph containing only the edge $e = \{1, 2\}$ is clearly unmatched. All graphs H that are unmatched after the first stage, and are not in M, have either S_1 or S_2 (as defined above) non-empty. Let G be constructed from H by deleting all edges within S_1 and S_2. Then $H \in [G, G']$, so H is matched in the second stage.

On the other hand, the graphs in M are unmatched in the first step and have empty S_1 and S_2, so they are unmatched by $\sim$. □

Proposition 5.2. *The matching $\sim$ above is a Γ-equivariant generalized Morse matching.*

Proof. We first show that $\sim$ is a generalized Morse matching.

Suppose for a contradiction that there were a closed cycle $v_1 \to v_2 \to \cdots \to v_n \to v_1$ in $P(\mathcal{N}_n)/\sim$. Its preimage in $P(\mathcal{N}_n)$ has the form

$$H_1 \supset G_2 \subseteq H_2 \supset \cdots \supset G_n \subseteq H_n \supset G_1 \subseteq H_1,$$

where G_i and H_i are in the same interval in $\sim$ (possibly, G_i and H_i are equal), and H_i and G_{i+1} differ by only one edge.

If for some i, H_i and G_i were matched in the first stage, we had $e = \{1, 2\} \in H_i$. If $G_{i+1} \neq G_i$, G_{i+1} must also contain e, and so be matched downwards, so $G_{i+1} = H_{i+1} = H_i \setminus f$ for some edge f. Inductively, we then see that all graphs in the sequence above contain e, and that the sequence is decreasing (and strictly decreasing every second step). This contradicts the fact that the sequence indeed was a cycle.

So $e \notin H_i$ for all i. So all matchings in the cycle are made in the second stage. Then, the "$\subseteq$" steps in the sequence do not affect S_i, v_i or the path from i to v_i for $i = 1$ or 2. This is immediate from the construction of $\sim$. So for the cycle to be closed, this must not be affected in the "$\supset$" steps either.

Hence, in the "$\supset$" steps, we are removing edges that are further away from v_i than S_i is. On the other hand, these edges are unaffected by what happens in the "$\subseteq$" steps. This means that such edges can only be removed when going to the right in the sequence above. But every "$\supset$" step is strict inclusion, since G_{i+1} is a proper face of H_i in $\mathcal{N}_n$.

This contradicts our assumption that the cycle was closed. Hence $P(\mathcal{N}_n)/\sim$ is acyclic, so $\sim$ is a generalized Morse matching.

Now the construction of $\sim$ was independent of the labelling of the nodes $3, \ldots, n$, and did not depend on which was which of 1 and 2. It follows that $\sim$ is Γ-equivariant. □

We can now conclude that there is a Morse complex $\mathcal{M}$ with one $(p-1)$-cell for each graph in M with p edges, such that $\mathcal{M} \simeq_\Gamma \mathcal{N}_n$. To make this result interesting, we only need to determine the structure of $\mathcal{M}$.

Apart from the graph with the only edge $\{1, 2\}$, all critical graphs consist of two disjoint chains, and thus have $n-2$ edges. We have already argued that there are $(n-1)!$ critical graphs of the second kind

So $\mathcal{M}$ has one 0-cell and $(n-1)!$ cells of dimension $(n-3)$, so $\mathcal{M} \cong \bigvee_{i=1}^{(n-1)!} S_i^{n-3}$. It only remains to determine the group action.

5.3. Group action on the Morse complex

If the two chains of G have the same length, there is exactly one element of Γ that fixes G, namely the one that interchanges the two chains. If the chains have different length, there is no group element fixing G, since such a group element must map the labels in the longest chain to themselves, and the labels in the shortest chain to themselves.

The orbit of a graph G is determined by the length of G's longest chain. This can be any number between $\frac{n}{2}$ and $n-1$, so there are $\lfloor \frac{n}{2} \rfloor$ orbits in $\mathcal{M}$. If the chain lengths in an orbit are different, the orbit has size $2(n-2)!$. This is because every graph in the orbit can be identified with a permutation of $3, \ldots, n$, together with a choice of which of the chains 1 should be in. If the two chain lengths are the same, we do not have this last choice, so then the orbit has size $(n-2)!$.

To state results about the Γ-homotopy type of $\mathcal{M}$ explicitly, we will need some (standard) notation:

Definition 5.1. Let (X, x_0) and (Y, y_0) be based spaces. Define the smash

$$X \wedge Y \stackrel{\text{def}}{=} X \times Y/(\{x_0\} \times Y \cup X \times \{y_0\}).$$

If Γ is a topological group, let Γ^+ be Γ together with a disjoint base point $+$, fixed by Γ. Then, $\Gamma^+ \wedge S^n$ is a wedge of spheres indexed by Γ.

If $H \subseteq \Gamma$ acts on X, then H acts on $\Gamma^+ \wedge X$ by $h[\gamma, x] = [\gamma h^{-1}, hx]$. Let $\Gamma^+ \wedge_H X$ denote $\Gamma^+ \wedge X$ modulo this group action. It inherits a group action of Γ on the first coordinate.

We see that, as Γ-complexes, $\Gamma^+ \wedge S^n$ can be constructed by attaching $\Gamma \times D^n$ to a 0-cell, and $\Gamma^+ \wedge_H S^n$ by attaching $\Gamma \times_H D^n$ to a 0-cell. Moreover, $\Gamma^+ \wedge S^n \cong \Gamma^+ \wedge_H S^n$ if H is trivial.

Looking back at the homotopy type of $\mathcal{N}_n$, we can distinguish two cases. The case where n is even gets pretty involved; we will carry out the details below. When n is odd, however, the classification becomes nice and simple, and we write down this as a theorem.

Theorem 5.3. *Let $n \geq 3$ be an odd number and let $\mathcal{N}_n$ and Γ be as defined above. Then*

$$\mathcal{N}_n \simeq_\Gamma \bigvee_{i=1}^{\frac{n-1}{2}} \Gamma^+ \wedge S_i^{n-3}.$$

The right-hand side denotes the wedge of $\frac{n-1}{2}$ orbits of $(n-3)$-cells, where the group action on each orbit is free. Note that, when the group action is ignored, we have $\frac{n-1}{2}|\Gamma| = (n-1)!$ top dimensional cells in the Morse complex.

In the case where n is even, we get another problem. Since one of the orbits has non-free group action, we cannot be sure a priori that the all the cells can be identified at one point. The following lemma will solve that problem.

Lemma 5.4. *Any two graphs in M, except the graph containing only $\{1, 2\}$, are unrelated in $P(\Sigma)/\sim$.*

Proof. Assume that the graphs G and H were critical, and that $G < H$ in $P(\Sigma)/\sim$. Then there were a sequence

$$H \supset G_1 \subseteq H_1 \supset \cdots \supset G_n \subseteq H_n \supset G$$

of graphs where G_i were matched to H_i by $\sim$. Since H is critical, hence the union of two disjoint chains, it follows that G_1 is the union of three disjoint chains. But such graphs are matched to $G_1 \cup e$ where $e = \{1, 2\}$. By induction, we see that all G_i either contain the edge e, or have at least three connected components (since $H_i \setminus \{e\}$ has at least three connected components). So as G is $\sim$-critical, it contains only the edge e.

This proves the lemma. □

So there are no relations between the graphs corresponding to $(n-3)$-cells in the Morse complex. Hence, they can show up in any order in the linear extension in the proof of Theorem 4.2. In particular, we can choose to save the graphs with two equally long chains to last. Let $\mathcal{M}'$ be the Morse complex as it stands before we have added these. In the rest of the paper, we assume n to be even.

The cells corresponding to graphs with equally long chains, have invariance groups of order two, transposing $\{1, 2\}$ and $\frac{n}{2} - 1$ other, pairwise disjoint, pairs. We let $\Lambda \subseteq \Gamma$ be the cyclic group of order two, generated by the permutation $\tau \stackrel{\text{def}}{=} (1, 2)(3, 4) \cdots (n-1, n)$.

Now

$$\mathcal{M} = \mathcal{M}' \cup_{\Gamma \times_{C_2} \partial D^{n-3}} \Gamma \times_{C_2} D^{n-3}.$$

The group action on each orbit in $\mathcal{M}'$ is free, so the only point in $\mathcal{M}'$ that is left invariant by some group element is the 0-cell. So for the boundary map to be Γ-equivariant, it must attach each of the last cells to the 0-cell in $\mathcal{M}'$. Thus, $\mathcal{N}_n$ is Γ-homotopic to a wedge of spheres also when n is even. This wedge is described in the following theorem.

Theorem 5.5. *Let $n > 3$ be an even number, and let $\mathcal{N}_n$ and Γ be as defined above. Then*

$$\mathcal{N}_n \simeq_\Gamma \bigvee_{i=1}^{\frac{n}{2}} \Gamma^+ \wedge_{H_i} S_i^{n-3},$$

where H_i is trivial for $i = 1, \ldots, \frac{n}{2} - 1$ and where $H_{\frac{n}{2}} = \Lambda = \langle \tau \rangle$.

5.4. Group action on homology

The action by Γ on $\mathcal{N}_n$ induces an action on the homology of $\mathcal{N}_n$. By Γ-invariant homotopy equivalence, the Γ-module structure on $\tilde{H}_*(\mathcal{N}_n)$ is equal to the same structure on $\tilde{H}_*(\mathcal{M})$.

n-1—···—3—1   2—4—···—n

Fig. 8. The critical graph fixed by τ.

As before, we need to separate the cases when n is even and odd. It is immediately clear that the reduced homology vanishes in all degrees except $n-3$.

The homology of the right-hand side in Theorem 5.3 is easy to calculate, componentwise. Indeed, $\Gamma^+ \wedge S_i^{n-3}$ is a wedge of spheres indexed and permuted by Γ. Hence $\tilde{H}_{n-3}(\Gamma^+ \wedge S_i^{n-3}) \cong \mathbb{Z}[\Gamma]$, where Γ acts by multiplication from the left. We get as an immediate consequence:

Theorem 5.6. *For odd numbers $n \geq 3$, we have $\tilde{H}_{n-3}(\mathcal{N}_n) = (\mathbb{Z}[\Gamma])^{\frac{n-1}{2}}$. For $i \neq n-3$, $\tilde{H}_i(\mathcal{N}_n) = 0$.*

In the even case, we only need to determine the homology of $\Gamma^+ \wedge_\Lambda S_i^{n-3}$ in degree $n-3$, where Λ is generated by $\tau = (1,2)(3,4)\cdots(n-1,n)$ as before. In general, $\tilde{H}_k(\Gamma^+ \wedge_\Lambda S^k) \equiv \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}$, where the Λ-module structure on the last factor is induced by the Λ-action on S^k. In our case, S^{n-3} corresponds to the critical $(n-3)$-simplex given by the graph in Fig. 8, with its boundary collapsed to a point.

Now τ acts on this simplex by transposing its vertices pairwise. So the action is a composition of $\frac{n}{2}-1$ reflections, and thus has degree $(-1)^{\frac{n}{2}-1}$.

Thus, if $n \equiv 2 \bmod 4$, then Λ acts trivially on the factor $\mathbb{Z}$, and if $n \equiv 0 \bmod 4$, then τ acts on $\mathbb{Z}$ by multiplication by -1.

As abelian groups (with Γ-action ignored for the moment), we have

$$\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z} \cong \mathbb{Z}[\Gamma/\Lambda] \cong \mathbb{Z}[S_{n-2}],$$

since each coset in Γ/Λ has one representative that does not permute 1 and 2. The factor S_{n-2} of $\Gamma \cong C_2 \times S_{n-2}$ acts naturally on the homology, by multiplication from the left in $\mathbb{Z}[S_{n-2}]$.

We only need to check what the C_2 factor of Γ does to the homology of $\mathcal{N}_n$. Let γ be the generator of this factor, let μ be an element of $\mathbb{Z}[S_{n-2}]$, and let $\nu \in S_{n-2}$ be the involution $(1,2)\cdots(n-3,n-2)$. From easy calculations (that get messy only because of the many isomorphisms involved), we see that multiplication by γ in $\tilde{H}_{n-3}(\mathcal{N}_n)$ induces an action on $\mathbb{Z}[S_{n-2}]$ by $\gamma\mu = \pm\mu\nu$. The sign is negative iff τ acts non-trivially on the homology.

Thus we can summarise the case where n is even as well.

Theorem 5.7. *For even numbers $n \geq 3$, we have $\tilde{H}_{n-3}(\mathcal{N}_n) = (\mathbb{Z}[\Gamma])^{\frac{n}{2}-1} \times \mathbb{Z}[\S_n - 2]$.*

Γ acts by multiplication from the left, except the factor C_2 of Γ, which acts on the factor $\mathbb{Z}[S_{n-2}]$ of $\tilde{H}_{n-3}(\mathcal{N}_n)$ by

$$\gamma^m\mu = \begin{cases} \mu\nu^m & \text{if } n \equiv 2 \bmod 4 \\ (-1)^m\mu\nu^m & \text{if } n \equiv 0 \bmod 4 \end{cases}$$

where $\nu = (1,2)\cdots(n-3,n-2)$.

For $i \neq n-3$, $\tilde{H}_i(\mathcal{N}_n) = 0$.

This concludes the analysis of the Γ-module structure on $\mathcal{N}_n$.

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